3 Yr. Degree/4 Yr. Honours 1st Semester Examination, 2023 (CCFUP)

Subject : Mathematics

Course : MATH1011 (MAJOR)

(Calculus, Geometry & Vector Calculus)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* of the following questions: 2×10=20

(a) Determine the asymptotes of $x^3 + y^3 = 3ax^2$.

(b) Evaluate $\lim\limits_{x\to 0} (\cos x)^{\cot^2 x}$.

(c) If $\lim\limits_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$ be finite, find the value of 'a' and the value of limit.

(d) If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that $(x^2-1)y_2 + xy_1 - m^2y = 0$.

(e) Evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x\, dx$.

(f) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$.

(g) Find the parametric equation of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(h) Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 5$ at (2, 0, 1).

(i) Find the equation of the cone whose vertex is the origin and the base is the circle $x = a, y^2 + z^2 = b^2$.

(j) Show that the semilatus rectum of a conic is the harmonic mean between the segments of a focal chord.

(k) Find the nature of the conicoid $3x^2 - 2y^2 - 12x - 12y - 6z = 0$.

(l) If $\vec{e}_1$ and $\vec{e}_2$ are unit vectors and θ be the angle between them, then show that $|\vec{e}_1 - \vec{e}_2| = 2\sin\frac{\theta}{2}$.

(m) Show that $[\vec{\alpha} + \vec{\beta}, \vec{\beta} + \vec{\gamma}, \vec{\gamma} + \vec{\alpha}] = 2\,[\vec{\alpha}, \vec{\beta}, \vec{\gamma}]$, where $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are any three vectors.

(n) Give an example to show that a continuous vector valued function may not be differentiable.

(o) If $\vec{r} = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + at\tan\alpha\,\hat{k}$, then find $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right]$.

2. Answer *any four* of the following questions: 5×4=20

(a) Reduce the equation $4x^2 + 4xy + y^2 - 4x - 2y + \lambda = 0$, to the canonical form and determine the nature of the conic for different values of λ.

(b) State and prove Leibnitz's theorem for nth derivative of the product of two functions of the same single variable.

(c) If $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ is the envelope of $\frac{x}{a} + \frac{y}{b} = 1$ where a, b are variable parameters and c is a constant, then prove that $a^2 + b^2 = c^2$.

(d) Determine the entire surface area formed by the revolution of the upper half of the cardioid $r = a\,(1 + \cos\theta)$ about the initial line.

(e) Find the equation and the type of the conic which passes through (–2, 0), touches the y-axis at the origin and has its centre at (1, 1).

(f) Show that the vector function $\vec{f}(t)$ has constant direction if and only if $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$. 2½x2

3. Answer *any two* of the following questions: 10×2=20

(a) (i) Find the all asymptotes of the curve $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$.

(ii) Show that the condition that the straight line $\frac{l}{r} = a\cos\theta + b\sin\theta$ may touch the circle $r = 2k\cos\theta$ is $b^2k^2 + 2ak = 1$. 5+5

(b) (i) If $I_{m,n} = \int \sin^m x \cos^n x\, dx$, where $m, n \in \mathbb{N} - \{1\}$, then show that $I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$.

(ii) Find the length of the arc of the parabola $y^2 = 16x$ measured from the vertex to an extremity of the latus rectum. 5+5

(c) (i) Reduce $6y^2 - 18yz - 6zx + 2xy - 9x + 5y - 5z + 2 = 0$ in canonical form and state the nature of the surface represented by it.

(ii) Show that the condition that the plane $ax + by + cz = 0$ may cut the cone $yz + zx + xy = 0$ in perpendicular lines is $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$. 5+5

(d) (i) Show that the vectors $\vec{a} \times (\vec{b} \times \vec{c}), \vec{b} \times (\vec{c} \times \vec{a}), \vec{c} \times (\vec{a} \times \vec{b})$ are coplanar.

(ii) Prove that curl grad $f = 0$ for scalar point function f.

(iii) Show that div curl $\vec{f} = 0$ for vector point function $\vec{f}$. 4+3+3

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